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OPTIMAL DESIGN OF MULTI-PURPOSE TIE COLUMN OF SOLID CONSTRUCTION

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Abstract—The paper considers the problem of maximizing the Euler buckling load of an elastic pin-ended member of solid construction and of given volume (mass) subject to a prescribed maximum permissible elongation when the member acts as a tie. The multi-purpose optimization not only unifies the design procedure of mass-produced structural/mechanical elements, but also provides a practically acceptable design in that the cross-sectional area of the optimal member does not vanish at any point along its length—a situation quite common in optimization for a single design requirement. However, it is shown that for cross sections of solid construction the constraint on elongation in tie action is weaker than that on maximum stress. The efficiency of optimal design is judged by comparing it with a prismatic member of the same volume (mass).

INTRODUCTION

A typical feature of the optimal design of elastic, pin-ended structural/mechanical elements of given material volume for a single design requirement (maximum stiffness[1], maximum fundamental frequency[2], maximum Euler buckling load[3]) is the vanishing of cross-sectional area at the simply supported ends. One way to overcome such an impractical design is to put a constraint on the minimum permissible area or the maximum permissible stress. The other is to design the member for more than one requirement such that the occurrence of vanishing cross sections is naturally avoided. One such additional design requirement is on the elongation of the member as a tie; the former is to be designed to have maximum Euler buckling load for a given volume (mass). Clearly, the member cannot act as a tie and a column at the same time, but at different times during its design life. Optimization of design of structural/mechanical elements to meet more than one design requirement (e.g. that of column and tie actions) is also advantageous from the point of rationalizing the design of these elements. Situations in which the members may have to act as a tie or as a column are frequently met in practice, particularly in the design of trusses under dynamic loading.

Problems of optimal design of multi-purpose structural/mechanical elements have received some attention. Thus, Prager and Shield[4] presented the minimum-weight design of a sandwich bar that is to act as a tie or as a beam. This approach for a sandwich bar was subsequently extended to bars of solid section[5]. Another type of minimum-weight optimization problem involving solid/hollow prismatic bars that have to serve as a beam or as a shaft was recently treated[6, 7].

The aim of this paper is to illustrate how an elastic, pin-ended member of solid construction and of given material volume (mass) can be designed to have the maximum Euler buckling load and a prescribed maximum permissible elongation due to axial tension. The member is to serve as a tie for a part of its design life and as a column for the rest.

It is not unusual in multi-purpose optimization that the optimal design be governed by only one of the two design requirements. Such a possibility has been examined in detail. In particular, it is shown that for cross sections of solid construction the constraint on maximum elongation in tie action above a certain value is weaker than that on the maximum stress in the sense that even with a zero cross-sectional area at the simply-supported ends the constraint on maximum elongation can be satisfied.

FORMULATION OF THE PROBLEM AND THE NECESSARY OPTIMALITY CONDITION

Consider an elastic member of length L, volume V pinned at one of its ends and supported on rollers at the other and subjected to an axial tension T^*. The longitudinal elongation $u(\xi)$ at a distance ξ from the pinned end satifies the following differential equation and boundary condition in non-dimensional form:

$$\alpha(x)u_x = T; \qquad u(o) = 0, \tag{1}$$

in which $\alpha(x) = A(\xi)L/V$ is the dimensionless cross-sectional area, $T = T^*L/EV$, E being the Young's modulus, and subscript x denotes differentiation with respect to the non-dimensional longitudinal coordinate $x = \xi/L$. The variation in $\alpha(x)$—assumed symmetric about the midspan, $x = \frac{1}{2}$—has to be such as to meet the following design requirement

$$u(1) = 2T\int_0^{1/2} \frac{1}{\alpha(x)}\,\mathrm{d}x \leq \lambda_o, \tag{2}$$

where $\lambda_o = \lambda_o^*/L$ is the specified maximum permissible elongation in non-dimensional form.

Now, if the elastic bar considered above be subjected at times during its design life to an axial compression P^* the lateral deflection $v(\xi)$ in elastic buckling should satisfy the following differential equation and boundary conditions

$$\alpha^n(x)v_{xx} + Pv = 0; \qquad v(o) = v_x(\tfrac{1}{2}) = 0, \tag{3}$$

where symmetry has been accounted for, and $P = P^*L^{2+n}/EcV^n$. Here it has been assumed that the second moment of area, $I(\xi)$, and the cross-sectional area, $A(\xi)$, of the member are related through

$$I(\xi) = cA^n(\xi),$$

where c and n are determined by the cross-sectional shape, $n = 1$ representing sandwich construction and $n = 2$, 3—solid construction. In this paper only the cross sections of *solid* construction are considered.

The variation in $\alpha(x)$ has to be such that, besides meeting the design requirement (2), the total volume of the member is a given constant value, V,

$$2\int_0^{1/2} \alpha(x)\,\mathrm{d}x = 1. \tag{4}$$

The optimization problem under consideration consists in determining the cross-sectional area variation along the length of the member that satisfies the differential equations (1) and (3) meet the requirements (2) and (4) and maximizes the Euler buckling load, P_{cr}, of the member

$$P_{cr} = \int_o^{1/2} \alpha^n(x)v_{xx}^2\,\mathrm{d}x \bigg/ \int_o^{1/2} v_x^2\,\mathrm{d}x \to \max. \tag{5}$$

To derive the necessary optimality condition for this optimization problem we write an auxiliary functional, Π, that includes the design constraints (2) and (4) through Lagrange multipliers β_1 and β_2

$$\Pi = \frac{\int_o^{1/2} \alpha^n v_{xx}^2\,\mathrm{d}x}{\int_o^{1/2} v_x^2\,\mathrm{d}x} + \beta_1\left(\frac{\lambda_o}{T} - 2\int_o^{1/2}\frac{\mathrm{d}x}{\alpha}\right) + \beta_2\left(1 - 2\int_o^{1/2}\alpha\,\mathrm{d}x\right).$$

Setting the first variation of Π with respect to α to zero, we get, after multiplying throughout by the constant $\int_o^{1/2} v_x^2\,\mathrm{d}x$ and introducing new multipliers μ and ν, the following necessary optimality condition:

$$\alpha^{n-1}v_{xx}^2 + \mu/\alpha^2 = \nu, \tag{6}$$

which, on substitution of v_{xx} from (3), reduces to

$$\alpha(x) = [\mu\alpha^{n-1} + P^2v^2)/\nu]^{1/(n+1)}. \tag{7}$$

Before describing briefly the iterative procedure used to arrive at the optimal design it is necessary to investigate the possible existence of optimal designs that are governed by only one of the two design requirements.

POSSIBLE SINGLE-PURPOSE OPTIMAL DESIGNS

As mentioned in Introduction it is quite possible in the optimal design for more than one design requirement that the design be governed by only one of the design requirements. In other words, it is possible that optimal tie/column design itself is the optimal design for both tie and column actions. To investigate such a possibility consider first the optimal tie problem—that of minimizing the longitudinal elongation of a member of solid construction of given volume.

The optimality condition for this special problem, which follows from the optimality condition for general problem formulated above, (6), is $\alpha(x) = (\mu/\nu)^{1/2} = \gamma^{1/2}$. The constant γ is evaluated from the isoperimetric condition (4), whereupon $\alpha(x) = 1$ and $\lambda_{\min}/T = 1$. The optimal tie for all values of n is, therefore of constant cross section along its length, and if used as a column will have a buckling load $P_{cr} = \pi^2$. Consequently, if the prescribed maximum permissible elongation (2) of the multi-purpose bar in tie action satisfies the inequality $\lambda_o/T \leq 1$, the prismatic bar itself will be the optimal design for both tie and column actions.

Next, consider the other possibility—that the optimal column design meets both the tie and column requirements. The optimal column design for $n = 2$ (geometrically similar cross sections) is available in literature[3]. From this solution it is known, for example, that the Euler buckling load of such a column is larger than that of a prismatic column of the same volume by about a third and that the optimal column area vanishes at the simply supported ends. For our purpose we also need to know the corresponding values for $n = 3$. Moreover, it is important to know the behaviour of the area function near the simply supported ends in order to be able to judge the longitudinal elongation, were the optimal column used as a tie.

To this end let us substitute for α from the optimality condition for this special problem [obtained by setting $\mu = 0$ in (6)] into the differential equation for column buckling (3), whereupon

$$-Pv\,v_{xx}^{(n+1)/(n-1)} = \text{const.} \tag{8}$$

The constant could be chosen as the normalizing factor because the solution of the homogeneous boundary value problem (3) is only known to within a constant multiplier. Moreover, let us assume that in the neighbourhood of $x = 0$, v can be expanded in a power series $v = Ax + Bx^m$ and try to find the lowest non-integer value of m. For (8) to be satisfied, it is easily shown that

$$m = (n+3)/(n+1). \tag{9}$$

In other words, v_{xx} has singularity of the type $x^{-[(n-1)/(n+1)]}$, i.e. of the type $x^{-1/3}$ and $x^{-1/2}$ for $n = 2$ and 3, respectively. More importantly, α tends to approach zero according to $x^{2/3}$ and $x^{1/2}$, respectively, as $x \to 0$. Thus, in the neighbourhood of $x = 0$

$$\begin{aligned} \alpha &\to x^{2/3}, \quad n = 2, \\ \alpha &\to x^{1/2}, \quad n = 3. \end{aligned} \tag{10}$$

Consequently, the optimal column for both $n = 2$ and 3 could well be used as a tie, provided the prescribed λ_o/T (2) value is greater than or equal to that of the optimal column. This is because, although $\alpha(o) = 0$, the variation in $\alpha(x)$ in the neighbourhood of $x = 0$ is such that $1/\alpha$ has an integrable singularity at $x = 0$. In fact, our numerical calculations show that when $n = 2$,

$P_{cr} = 13.16$ (an improvement of 33.3% over that of a prismatic column of the same volume) and $\lambda/T = 1.295$ and for $n = 3$, $P_{cr} = 13.88$ (an improvement of 40.7% over that of a prismatic bar of the same volume) and $\lambda/T = 1.137$.

Therefore, the optimal design meeting both design constraints is to be sought only if the constraint on the longitudinal elongation (2) satisfies the inequalities

$$\begin{aligned} &1 < \lambda_o/T < 1.295, \quad n = 2 \\ &1 < \lambda_o/T < 1.137, \quad n = 3. \end{aligned} \tag{11}$$

The second inequality (for $n = 3$) is depicted graphically in Fig. 1.

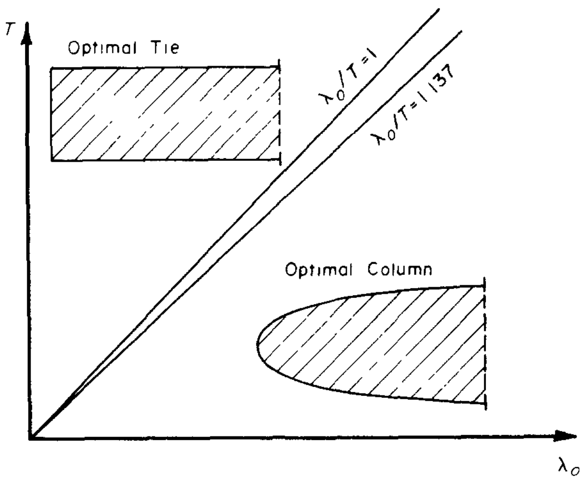

Fig. 1. Possible regions of single-variable optimal designs. $n = 3$

The iterative procedure used to solve the optimal column problem is briefly described in Appendix.

In connection with the two inequalities (11) it is perhaps justified to reemphasize the remarks made in Introduction concerning the weakness of the constraint on longitudinal elongation in comparison with the stronger constraint on the maximum allowable stress. The latter could not be satisfied by the optimal column design for any value of n.

SOLUTION OF THE MULTI-PURPOSE OPTIMIZATION PROBLEM AND CONCLUSIONS

If the prescribed maximum permissible elongation (2) satisfies the inequality (11) corresponding to the value of n, the optimal design will be governed by both tie and column actions. In this general case the optimality condition (7) has to be solved together with the boundary value problem (3) and the design constraints (2) and (4) to arrive at $\alpha(x)$ and P_{cr}.

The unknown constants μ and ν appearing in the optimality condition (7) are to be found from the design constraint (2) and the isoperimetric condition (4). The latter reduces to

$$(\nu)^{1/(n+1)} = 2\int_o^{1/2} (\mu\alpha^{n-1} + P^2 v^2)^{1/(n+1)}\,\mathrm{d}x. \tag{12}$$

However, the design constraint (2) does not provide an explicit expression for μ. Therefore, it was found necessary to calculate μ in an inverse manner, whereby, instead of λ_o/T, $\mu > 0$ was specified and ν evaluated from (12). The corresponding "given" value of λ_o/T was subsequently calculated from (2). It should be noted that $\mu = 0$ corresponds to optimal column design.

For a given value of $\mu > 0$ the iterative procedure consisted in making an initial guess for $(v_{xx})_i$ not necessarily satisfying the boundary conditions (3). In fact, in the first iteration $(i = 1)v_{xx}$ was chosen to be identically equal to unity. The subsequent steps in the iterative procedure were

(i) $$(v_x)_i = -\int_x^{1/2} (v_{\eta\eta})_i \, d\eta.$$

(ii) $$(v)_i = \int_o^{x} (v_\eta)_i \, d\eta.$$

(iii) Assume $\alpha_i(x) \equiv 1$, when $i = 1$.

(iv) $$(P_{cr})_i = \int_o^{1/2} \alpha_i^n (v_{xx}^2)_i \, dx \Big/ \int_o^{1/2} (v_x^2)_i \, dx.$$

(v) $$(\nu)_i^{1/(n+1)} = 2\int_o^{1/2} [\mu(\alpha)_i^{n-1} + (P)_i^2 (v^2)_i]^{1/(n+1)} \, dx.$$

(vi) $$\alpha_i(x) = [\beta\alpha_i^{n-1} + (P)_i^2 (v^2)_i]^{1/(n+1)} / (\nu)_i^{1/(n+1)}.$$

(vii) Repeat steps (iv)–(vi) within the inner loop. The results of the inner iteration loop are α_i, ν_i, P_i for the assumed value of $(v_{xx})_i$.

(viii) $$(v_{xx})_{i+1} = -P_i v_i / \alpha_i^n$$

(ix) $$(\lambda/T)_i = 2\int_o^{1/2} \frac{1}{\alpha_i} \, dx.$$

(x) Compare $(\lambda/T)_i$ with $(\lambda/T)_{i-1}$ for $i > 1$. Repeat steps (i)–(ix), if error greater than a prescribed upper limit.

(xi) Repeat steps (i)–(x) for different values of μ.

The variation of $\alpha(x)$ for various "given" values of λ_o/T in the range (12) is shown in Figs. 2 and 3 for $n = 2$, respectively. For $n = 2$, the linear dimension of cross section ($\alpha^{1/2}$) rather than the cross-sectional area is shown. For comparison the limiting cases of optimal tie ($\lambda_o/T = 1$) and the optimal column are also shown.

To get an idea of the economy achieved by optimization the Euler buckling load of the optimal tie-column is compared with that of a prismatic bar of the same volume. The percentage gain in the buckling load is shown in Figs. 4 and 5 for $n = 2$ and 3, respectively.

From Figs. 4 and 5 it is clear that substantial gain is possible by optimization. More

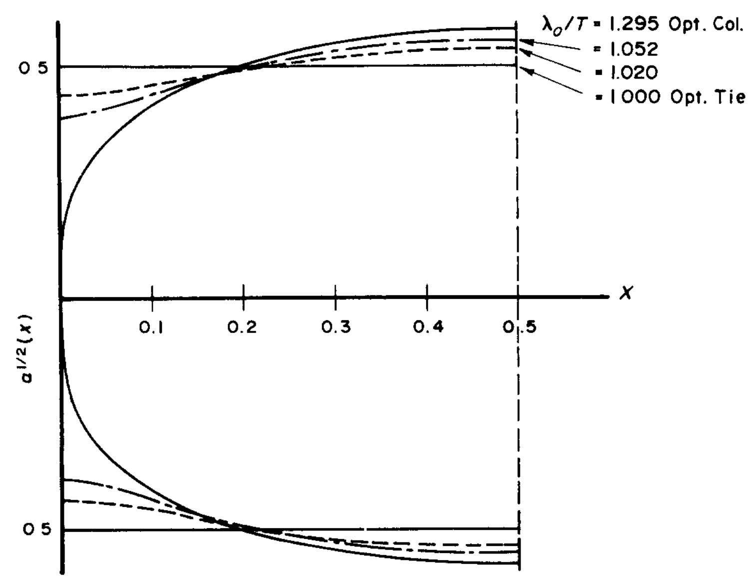

Fig. 2. Variation of the linear dimension of cross section ($\alpha^{1/2}$) for various values of λ_o/T, including those corresponding to single-variable optimal designs. $n = 2$.

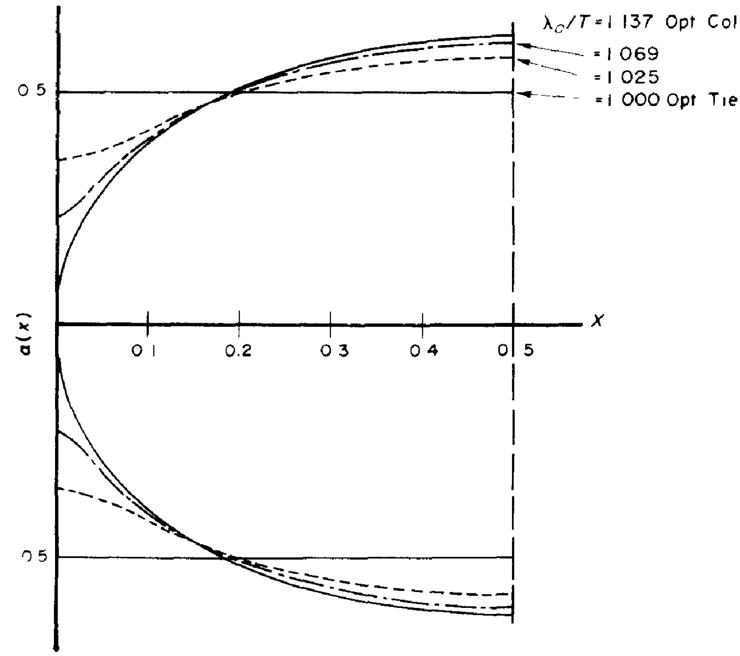

Fig. 3. Variation of the cross-sectional area for various values of λ_0/T including those corresponding to single-variable optimal designs. $n = 3$

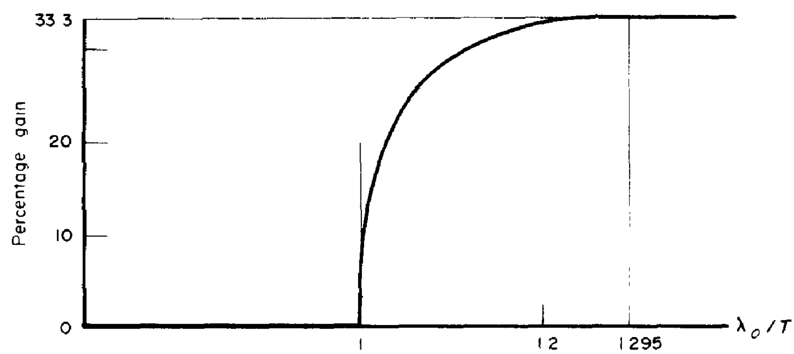

Fig. 4. Percentage increase in Euler buckling load over that of a prismatic column of the same volume for $n = 2$. Note the maximum value 33.3% corresponds to the optimal column design.

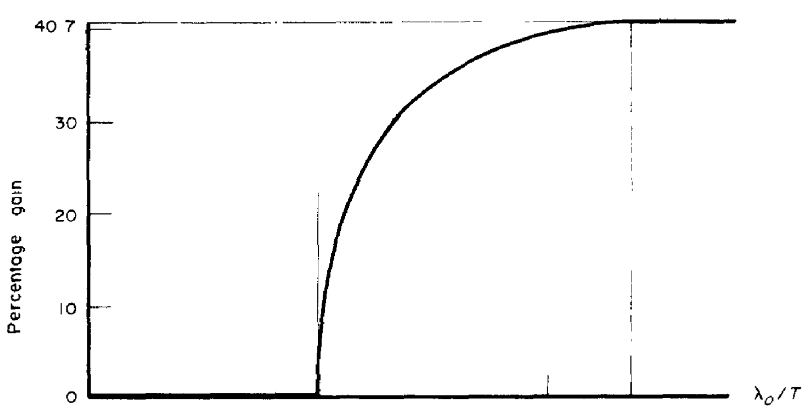

Fig. 5. Percentage increase in Euler buckling load over that of a prismatic column of the same volume for $n = 3$. Note the maximum gain 40.7% is possible at a finite value of λ_0/T—corresponding to optimal column

importantly, the optimal design is practically viable in that nowhere along the length of the member does the cross-sectional area reduce to an impractical zero value.

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APPENDIX

The iterative procedure used to solve the optimal column for $n = 3$ is described below (the accuracy was checked against known results for $n = 2$).

1. Assume a regular function $g_i(x) = x^{(n-1)/(n+1)}(v_{xx})_i$ arbitrarily in the first iteration ($i = 1$).

2. $$(v_x)_i = -\int_x^{1/2} (\eta)^{-[(n-1)/(n+1)]} g_i(\eta)\,\mathrm{d}\eta.$$

3. $$(v)_i = \int_o^x (v_\eta)_i\,\mathrm{d}\eta.$$

4. $$\alpha_i(x) = \frac{x^{2/(n+1)}}{g_i^{2/(n-1)}}.$$

5. Normalise $\alpha_i(x)$ using $2\int_o^{1/2} \alpha_i(x)\,\mathrm{d}x = 1$.

6. $$P_i = \int_o^{1/2} \alpha_i^n (v_{xx}^2)_i\,\mathrm{d}x \Big/ \int_o^{1/2} (v_x^2)_i\,\mathrm{d}x$$
$$= \int_o^{1/2} \alpha_i\,\mathrm{d}x \Big/ \int_o^{1/2} (v_x^2)_i\,\mathrm{d}x = 1 \Big/ 2\int_o^{1/2} (v_x)_i^2\,\mathrm{d}x.$$

7. $$g_{i+1(x)} = -\frac{x^{(n-1)/(n+1)} P_i v_i}{\alpha_i^n} = \frac{x^{(n-1)/(n+1)}}{(-P_i v_i)^{(n-1)/(n+1)}}.$$

8. Repeat steps (2)–(7) until the error is less than the permissible value.

9. $$\lambda/T = 2\int_o^{1/2} \frac{\mathrm{d}x}{\alpha(x)}.$$

Note at the lower limit the integral vanishes.